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ON THE USE OF 6-PARAMETER MULTILAYERED SHELL MODELS IN STRUCTURAL MECHANICS

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Abstract: The new geometrically exact multilayered shell models are considered. These models are based on the objective strain-displacement relationships represented in the local curvilinear coordinates and, therefore, may be used for the formulation of effective curved multilayered shell elements. However, the practical use of such elements require the development of constitutive equations, in order to overcome Poisson thickness and volumetric locking phenomena. For this purpose three types of the material stiffness matrix are studied.

1 Introduction

One of the main requirements of a finite element that is intended for the general analysis of shells is that it must lead to strain-free modes for rigid-body motions. The adequate representation of rigid-body motions is a necessary condition if an element is to have good accuracy and convergence properties. Therefore, when an inconsistent shell theory is used to construct any finite element, erroneous straining modes under rigid-body motions may appear. This problem has been only studied for the Kirchhoff-Love shell theory [1–3] and Timoshenko-Mindlin shell theory [4, 5]. Herein, the more general study on the basis of the first-order multilayered shell theory [6] is considered. As unknown functions six displacements of the bottom and top surfaces of the shell are selected.

It is common knowledge that in some works developing the solid-shell concept [7–9] displacement vectors of the face surfaces are also used and represented in some global Cartesian basis in order to exactly describe rigid-body motions. But in our first-order shell theory selecting as unknowns the displacements of face surfaces of the shell has a principally another mechanical sense and allows us to formulate any curved shell elements on the basis of strain-displacement relationships that are objective, i.e., invariant under all rigid-body motions. In order to circumvent thickness locking the modified laminate stiffness matrix [10, 11] and simplified material stiffness matrices symmetric [5, 7, 8, 12] or non-symmetric [13, 14] corresponding to the generalized plane stress state are employed.

2 Problem formulation

Consider a shell built up in the general case by the arbitrary superposition across the wall thickness of N layers of uniform thickness h_k. The kth layer may be defined as a 3D body of volume V_k bounded by two surfaces S_{k-1} and S_k, located at the distances δ_{k-1} and δ_k measured with respect to the reference surface S, and the edge boundary surface Ω_k (Fig. 1). The full edge boundary surface $\Omega = \Omega_1 + \Omega_2 + \ldots + \Omega_N$ is generated by the normals to the reference surface along the bounding curve $\Gamma \subset S$ (with the arc length s) of this surface. It is also assumed that the bounding surfaces S_{k-1} and S_k are continuous, sufficiently smooth and without any singularities. Let the reference surface S be referred to an orthogonal curvilinear coordinate system α_1 and α_2, which coincides with the lines of principal curvatures of its surface; $\mathbf{e}_1$ and $\mathbf{e}_2$ are the tangent unit vectors to the lines of principal curvatures; A_α and k_α are the Lamé coefficients and principal curvatures of the reference surface. The α_3 – axis is oriented along the unit vector $\mathbf{e}_3$ normal to the reference surface.

The constituent layers of the shell are supposed to be rigidly joined, so that no slip on contact surfaces and no separation of layers can occur. The material of each constituent layer is assumed to be linearly elastic, anisotropic, homogeneous or fiber reinforced, such that in each point there is a single surface of elastic symmetry parallel to the reference surface. Let p_i^- and p_i^+ be the intensities of the external loading acting on the bottom surface $S^- = S_0$ and top surface $S^+ = S_N$ in the α_i coordinate directions, respectively; $\mathbf{q}^{(k)} = q_\nu^{(k)}\mathbf{v} + q_t^{(k)}\mathbf{t} + q_3^{(k)}\mathbf{e}_3$ are the external loading vector acting on the edge boundary surface Ω_k, where $q_\nu^{(k)}$, $q_t^{(k)}$ and $q_3^{(k)}$ are the components of its vector in the ν, t and α_3 directions; $\mathbf{v}$ and $\mathbf{t}$ are the normal and tangential unit vectors to the bounding curve Γ. Here and in the following developments the index $k = \overline{1,\ N}$

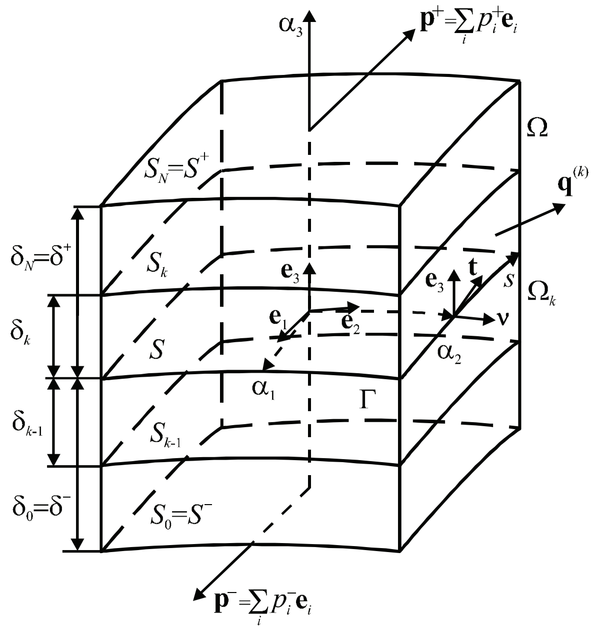

Fig. 1 Multilayered shell

identifies the belonging of any quantity to the kth layer; the abbreviation $(\,)_{,\alpha}$ implies the partial derivatives with respect to the coordinate α_1 and α_2; indices i, j, ℓ, m take the values 1, 2 and 3; Greek indices α, β, γ, δ take the values 1 and 2.

3 Shell kinematics

The first-order multilayered shell theory is based on the linear approximation of displacements in the thickness direction [13, 15]

$$\mathbf{u} = N^-(\alpha_3)\mathbf{v}^- + N^+(\alpha_3)\mathbf{v}^+, \tag{1a}$$

$$\mathbf{u} = \sum_i u_i \mathbf{e}_i, \quad \mathbf{v}^\pm = \sum_i v_i^\pm \mathbf{e}_i, \tag{1b}$$

$$N^-(\alpha_3) = \frac{1}{h}\left(\delta^+ - \alpha_3\right), \quad N^+(\alpha_3) = \frac{1}{h}\left(\alpha_3 - \delta^-\right), \tag{1c}$$

where $\mathbf{u}$ is the displacement vector; $u_i(\alpha_1,\ \alpha_2,\ \alpha_3)$ are the components of this vector; $\mathbf{v}^\pm$ are the displacement vectors of surfaces $S^\pm$; $v_i^\pm(\alpha_1,\ \alpha_2)$ are the components of these vectors; $N^\pm(\alpha_3)$ are the linear through-the-thickness shape functions; h is the thickness of the shell. It is important that displacement vectors (1b) are represented in the local reference surface frame $\mathbf{e}_i$ that allows one to reduce the costly numerical integration by deriving the stiffness matrix.

Substituting displacements (1a) into a vector form of the 3D strain-displacement relationships [5] and replacing Lamé coefficients by their values on the bottom and top surfaces $A_\alpha^\pm$ and middle surface $\overline{A}_\alpha$ in corresponding expressions for the in-plane and transverse shear components, the following equations are obtained

$$\varepsilon_{\alpha i} = N^-(\alpha_3)e_{\alpha i}^- + N^+(\alpha_3)e_{\alpha i}^+, \quad \varepsilon_{33} = e_{33}, \tag{2}$$

where $e_{\alpha\beta}^\pm$ and $e_{\alpha 3}^\pm$ are the in-plane and transverse shear components of the strain tensor of face surfaces $S^\pm$ defined by

$$2e_{\alpha\beta}^\pm = \frac{1}{A_\alpha^\pm}\mathbf{v}_{,\alpha}^\pm \cdot \mathbf{e}_\beta + \frac{1}{A_\beta^\pm}\mathbf{v}_{,\beta}^\pm \cdot \mathbf{e}_\alpha, \tag{3a}$$

$$2e_{\alpha 3}^\pm = \frac{\zeta_\alpha^\pm}{\overline{\zeta}_\alpha}\boldsymbol{\beta}\cdot\mathbf{e}_\alpha + \frac{1}{\overline{A}_\alpha}\mathbf{v}_{,\alpha}^\pm \cdot \mathbf{e}_3, \quad e_{33} = \boldsymbol{\beta}\cdot\mathbf{e}_3, \tag{3b}$$

$$\boldsymbol{\beta} = \frac{1}{h}\left(\mathbf{v}^+ - \mathbf{v}^-\right), \quad A_\alpha^\pm = A_\alpha\zeta_\alpha^\pm, \quad \overline{A}_\alpha = A_\alpha\overline{\zeta}_\alpha, \tag{3c}$$

$$\zeta_\alpha^\pm = 1 + k_\alpha\delta^\pm, \quad \overline{\zeta}_\alpha = 1 + k_\alpha\overline{\delta}, \quad \overline{\delta} = \frac{1}{2}\left(\delta^- + \delta^+\right).$$

Note that equations for the transverse shear strains (2) and (3b) differ from similar equations [5] and are more convenient for the finite element implementation. Strain-displacement relationships (2) and (3) are very attractive because they are objective, i.e., invariant under rigid-body motions. This may be readily proved by using a technique [5].

4 Hu-Washizu variational equation

The first-order multilayered shell theory developed is based on the assumed approximations of displacements (1) and displacement-dependent strains (2) in the thickness direction. Additionally, one should adopt the similar approximation for the assumed displacement-independent strains

$$\varepsilon_{\alpha i}^{\mathrm{AS}} = N^{-}\left(\alpha_3\right)E_{\alpha i}^{-} + N^{+}\left(\alpha_3\right)E_{\alpha i}^{+}, \quad \varepsilon_{33}^{\mathrm{AS}} = E_{33}. \tag{4}$$

Substituting approximations (1), (2) and (4) into the Hu-Washizu mixed variational principle [16] and accounting for that metrics of all surfaces parallel to the reference surface are identical and equal to the metric of the middle surface, one can derive

$$\iint_{\bar{S}}\left[\left(\mathbf{H}-\mathbf{DE}\right)^{\mathrm{T}}\delta\mathbf{E}+\left(\mathbf{E}-\mathbf{e}\right)^{\mathrm{T}}\delta\mathbf{H}-\mathbf{H}^{\mathrm{T}}\delta\mathbf{e}+\mathbf{P}^{\mathrm{T}}\delta\mathbf{v}\right]\bar{A}_1\bar{A}_2\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2 + \tag{5}$$

$$+\oint_{\bar{\Gamma}} H_{\Gamma}^{\mathrm{T}}\,\delta\mathbf{v}_{\Gamma}\left(1+k_N\bar{\delta}\right)\mathrm{d}s = 0.$$

Here, matrix notations are introduced

$$\mathbf{D} = \begin{bmatrix}
D_{1111}^{00} & D_{1111}^{01} & D_{1122}^{00} & D_{1122}^{01} & D_{1112}^{00} & D_{1112}^{01} & 0 & 0 & 0 & 0 & D_{1133}^{-} \\
D_{1111}^{01} & D_{1111}^{11} & D_{1122}^{01} & D_{1122}^{11} & D_{1112}^{01} & D_{1112}^{11} & 0 & 0 & 0 & 0 & D_{1133}^{+} \\
D_{2211}^{00} & D_{2211}^{01} & D_{2222}^{00} & D_{2222}^{01} & D_{2212}^{00} & D_{2212}^{01} & 0 & 0 & 0 & 0 & D_{2233}^{-} \\
D_{2211}^{01} & D_{2211}^{11} & D_{2222}^{01} & D_{2222}^{11} & D_{2212}^{01} & D_{2212}^{11} & 0 & 0 & 0 & 0 & D_{2233}^{+} \\
D_{1211}^{00} & D_{1211}^{01} & D_{1222}^{00} & D_{1222}^{01} & D_{1212}^{00} & D_{1212}^{01} & 0 & 0 & 0 & 0 & D_{1233}^{-} \\
D_{1211}^{01} & D_{1211}^{11} & D_{1222}^{01} & D_{1222}^{11} & D_{1212}^{01} & D_{1212}^{11} & 0 & 0 & 0 & 0 & D_{1233}^{+} \\
0 & 0 & 0 & 0 & 0 & 0 & D_{1313}^{00} & D_{1313}^{01} & D_{1323}^{00} & D_{1323}^{01} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D_{1313}^{01} & D_{1313}^{11} & D_{1323}^{01} & D_{1323}^{11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D_{2313}^{00} & D_{2313}^{01} & D_{2323}^{00} & D_{2323}^{01} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D_{2313}^{01} & D_{2313}^{11} & D_{2323}^{01} & D_{2323}^{11} & 0 \\
D_{3311}^{-} & D_{3311}^{+} & D_{3322}^{-} & D_{3322}^{+} & D_{3312}^{-} & D_{3312}^{+} & 0 & 0 & 0 & 0 & D_{3333}
\end{bmatrix}, \tag{6}$$

$$\mathbf{v} = \left[v_1^{-}\ v_1^{+}\ v_2^{-}\ v_2^{+}\ v_3^{-}\ v_3^{+}\right]^{\mathrm{T}}, \quad \mathbf{v}_{\Gamma} = \left[v_{\nu}^{-}\ v_{\nu}^{+}\ v_t^{-}\ v_t^{+}\ v_3^{-}\ v_3^{+}\right]^{\mathrm{T}},$$

$$\mathbf{e} = \left[e_{11}^{-}\ e_{11}^{+}\ e_{22}^{-}\ e_{22}^{+}\ 2e_{12}^{-}\ 2e_{12}^{+}\ 2e_{13}^{-}\ 2e_{13}^{+}\ 2e_{23}^{-}\ 2e_{23}^{+}\ e_{33}\right]^{\mathrm{T}},$$

$$\mathbf{E} = \left[E_{11}^{-}\ E_{11}^{+}\ E_{22}^{-}\ E_{22}^{+}\ 2E_{12}^{-}\ 2E_{12}^{+}\ 2E_{13}^{-}\ 2E_{13}^{+}\ 2E_{23}^{-}\ 2E_{23}^{+}\ E_{33}\right]^{\mathrm{T}},$$

$$\mathbf{H} = \left[H_{11}^{-}\ H_{11}^{+}\ H_{22}^{-}\ H_{22}^{+}\ H_{12}^{-}\ H_{12}^{+}\ H_{13}^{-}\ H_{13}^{+}\ H_{23}^{-}\ H_{23}^{+}\ H_{33}\right]^{\mathrm{T}},$$

$$\hat{\mathbf{H}}_{\Gamma} = \left[\hat{H}_{\nu\nu}^{-}\ \hat{H}_{\nu\nu}^{+}\ \hat{H}_{\nu t}^{-}\ \hat{H}_{\nu t}^{+}\ \hat{H}_{\nu 3}^{-}\ \hat{H}_{\nu 3}^{+}\right]^{\mathrm{T}}, \quad \mathbf{P} = \left[-p_1^{-}\ p_1^{+}\ -p_2^{-}\ p_2^{+}\ -p_3^{-}\ p_3^{+}\right]^{\mathrm{T}},$$

where **D** is the constitutive stiffness matrix whose components are defined in the next section; $v_\nu^\pm$, $v_t^\pm$ and $v_3^\pm$ are the components of displacement vectors of face surfaces in the coordinate system ν, t and α_3 (Fig. 1); k_N is the normal curvature of the reference bounding curve Γ; $\overline{\Gamma}$ is the bounding curve belonging to the middle surface $\overline{S}$; $H_{\alpha\beta}^\pm$, $H_{\alpha 3}^\pm$ and H_{33} are the stress resultants; $\hat{H}_{\nu\nu}^\pm$, $\hat{H}_{\nu t}^\pm$ and $\hat{H}_{\nu 3}^\pm$ are the external load resultants defined as

$$H_{\alpha i}^\pm = \sum_k \int_{\delta_{k-1}}^{\delta_k} \sigma_{\alpha i}^{(k)} N^\pm(\alpha_3)\,\mathrm{d}\alpha_3, \quad H_{33} = \sum_k \int_{\delta_{k-1}}^{\delta_k} \sigma_{33}^{(k)}\,\mathrm{d}\alpha_3, \tag{7}$$

$$H_{\nu æ}^\pm = \sum_k \int_{\delta_{k-1}}^{\delta_k} q_{æ}^{(k)} N^\pm(\alpha_3)\,\mathrm{d}\alpha_3 \quad (æ = \nu,\ t \text{ and } 3).$$

Mixed variational equation (5) may be used for constructing non-conventional assumed stress-strain four-node curved shell elements.

5 Constitutive equations

In this section four types of the constitutive equations are discussed. We consider first an orthotropic ply and then study the more general case of monoclinic symmetry.

5.1 Complete constitutive equations

Consider the *k*th orthotropic layer of the shell and denote its axes of symmetry as $\alpha_{1'}^{(k)}$, $\alpha_{2'}^{(k)}$ and α_3. In these axes of symmetry the equations of the complete 3D Hooke's law will be

$$\begin{gathered}
\varepsilon_{1'1'} = \frac{1}{E_1^{(k)}}\sigma_{1'1'}^{(k)} - \frac{\nu_{21}^{(k)}}{E_2^{(k)}}\sigma_{2'2'}^{(k)} - \frac{\nu_{31}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)},\\
\varepsilon_{2'2'} = -\frac{\nu_{12}^{(k)}}{E_1^{(k)}}\sigma_{1'1'}^{(k)} + \frac{1}{E_2^{(k)}}\sigma_{2'2'}^{(k)} - \frac{\nu_{32}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)},\\
2\varepsilon_{1'2'} = \frac{1}{G_{12}^{(k)}}\sigma_{1'2'}^{(k)}, \quad 2\varepsilon_{1'3} = \frac{1}{G_{13}^{(k)}}\sigma_{1'3}^{(k)}, \quad 2\varepsilon_{2'3} = \frac{1}{G_{23}^{(k)}}\sigma_{2'3}^{(k)},\\
\varepsilon_{33} = -\frac{\nu_{13}^{(k)}}{E_1^{(k)}}\sigma_{1'1'}^{(k)} - \frac{\nu_{23}^{(k)}}{E_2^{(k)}}\sigma_{2'2'}^{(k)} + \frac{1}{E_3^{(k)}}\sigma_{33}^{(k)},
\end{gathered} \tag{8}$$

where $\sigma_{1'1'}^{(k)}$, $\sigma_{2'2'}^{(k)}$, $\sigma_{33}^{(k)}$ and $\sigma_{1'2'}^{(k)}$ $\sigma_{1'3}^{(k)}$, $\sigma_{2'3}^{(k)}$ are the normal and shear components of the stress tensor in the $\alpha_{1'}^{(k)}$, $\alpha_{2'}^{(k)}$ and α_3 coordinate system; $E_1^{(k)}$, $E_2^{(k)}$ and $E_3^{(k)}$ are the elastic moduli; $G_{12}^{(k)}$, $G_{13}^{(k)}$ and $G_{23}^{(k)}$ are the shear moduli; $\nu_{ij}^{(k)}$ are Poisson's ratios. From reasons of symmetry, we have

$$\nu_{ij}^{(k)} E_j^{(k)} = \nu_{ji}^{(k)} E_i^{(k)} \quad \text{for} \quad i \neq j.$$

In coordinate directions α_1, α_2 and α_3, when a case of monoclinic symmetry is realized, the equations of the 3D Hooke's law can be represented in the more general form

$$\varepsilon_{ij} = \sum_{\ell,m} A^{(k)}_{ij\ell m} \sigma^{(k)}_{\ell m}, \tag{9}$$

where $A^{(k)}_{ij\ell m}$ are the components of the material tensor of the kth layer depending on engineering constants $E^{(k)}_i$, $G^{(k)}_{ij}$ and $\nu^{(k)}_{ij}$ whose expressions can be found in many books (see e.g. [16]) and are not displayed here.

Solving first three equations (9) for the in-plane stresses, one can find

$$\sigma^{(k)}_{\alpha\beta} = \sum_{\gamma,\delta} Q^{(k)}_{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} - \mu^{(k)}_{\alpha\beta 33} \sigma^{(k)}_{33}, \tag{10}$$

where

$$\mu^{(k)}_{\alpha\beta 33} = \sum_{\gamma,\delta} Q^{(k)}_{\alpha\beta\gamma\delta} A^{(k)}_{\gamma\delta 33}, \tag{11}$$

$$Q^{(k)}_{1111} = \frac{1}{\Delta_k}\left(A^{(k)}_{2222} A^{(k)}_{1212} - A^{(k)}_{2212} A^{(k)}_{1222}\right), \qquad Q^{(k)}_{1122} = \frac{1}{\Delta_k}\left(A^{(k)}_{1112} A^{(k)}_{1222} - A^{(k)}_{1122} A^{(k)}_{1212}\right),$$

$$Q^{(k)}_{1112} = \frac{1}{2\Delta_k}\left(A^{(k)}_{1122} A^{(k)}_{2212} - A^{(k)}_{1112} A^{(k)}_{2222}\right), \qquad Q^{(k)}_{2222} = \frac{1}{\Delta_k}\left(A^{(k)}_{1111} A^{(k)}_{1212} - A^{(k)}_{1112} A^{(k)}_{1211}\right),$$

$$Q^{(k)}_{2212} = \frac{1}{2\Delta_k}\left(A^{(k)}_{1112} A^{(k)}_{2211} - A^{(k)}_{1111} A^{(k)}_{2212}\right), \qquad Q^{(k)}_{1212} = \frac{1}{4\Delta_k}\left(A^{(k)}_{1111} A^{(k)}_{2222} - A^{(k)}_{1122} A^{(k)}_{2211}\right),$$

$$\Delta_k = A^{(k)}_{1211}\left(A^{(k)}_{1122} A^{(k)}_{2212} - A^{(k)}_{1112} A^{(k)}_{2222}\right) + A^{(k)}_{1222}\left(A^{(k)}_{1112} A^{(k)}_{2211} - A^{(k)}_{1111} A^{(k)}_{2212}\right) +$$

$$+ A^{(k)}_{1212}\left(A^{(k)}_{1111} A^{(k)}_{2222} - A^{(k)}_{1122} A^{(k)}_{2211}\right).$$

Substituting in-plane stresses (10) in the last equation (9) and solving for the transverse normal stress, one obtains

$$\sigma^{(k)}_{33} = \sum_{\gamma,\delta} C^{(k)}_{33\gamma\delta} \varepsilon_{\gamma\delta} + C^{(k)}_{3333} \varepsilon_{33}, \tag{12}$$

where

$$C^{(k)}_{33\alpha\beta} = -\frac{1}{\Lambda_k} \mu^{(k)}_{33\alpha\beta}, \qquad C^{(k)}_{3333} = \frac{1}{\Lambda_k}, \tag{13}$$

$$\mu^{(k)}_{33\alpha\beta} = \sum_{\gamma,\delta} Q^{(k)}_{\gamma\delta\alpha\beta} A^{(k)}_{33\gamma\delta}, \qquad \Lambda_k = -\sum_{\gamma,\delta} A^{(k)}_{33\gamma\delta} \mu^{(k)}_{\gamma\delta 33} + A^{(k)}_{3333}.$$

By using the transverse normal stress (12) into formula (10) yields

$$\sigma^{(k)}_{\alpha\beta} = \sum_{\gamma,\delta} C^{(k)}_{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} + C^{(k)}_{\alpha\beta 33} \varepsilon_{33}, \tag{14}$$

where

$$C^{(k)}_{\alpha\beta\gamma\delta} = Q^{(k)}_{\alpha\beta\gamma\delta} + \frac{1}{\Lambda_k} \mu^{(k)}_{\alpha\beta 33} \mu^{(k)}_{33\gamma\delta}, \qquad C^{(k)}_{\alpha\beta 33} = -\frac{1}{\Lambda_k} \mu^{(k)}_{\alpha\beta 33}. \tag{15}$$

Finally, one can derive from Hooke's law (9) the remaining equations for the transverse shear stresses as

$$\sigma_{\alpha 3}^{(k)} = 2\sum_{\gamma} C_{\alpha 3\gamma 3}^{(k)} \varepsilon_{\gamma 3}, \tag{16}$$

where

$$C_{1313}^{(k)} = \frac{1}{d_k} A_{2323}^{(k)}, \qquad C_{1323}^{(k)} = -\frac{1}{d_k} A_{1323}^{(k)}, \qquad C_{2323}^{(k)} = \frac{1}{d_k} A_{1313}^{(k)}, \tag{17}$$
$$d_k = 4\left(A_{1313}^{(k)} A_{2323}^{(k)} - A_{1323}^{(k)} A_{2313}^{(k)} \right) .$$

Unfortunately, such shell formulation on the basis of the complete 3D constitutive law (12)-(17) is deficient because so-called thickness locking [9, 17] can occur. This phenomenon occurs in bending dominated shell problems when Poisson's ratios are not equal to zero. In order to avoid thickness locking the effective remedies may be used.

5.2 Modified constitutive equations

It is well-known that in the first-order multilayered shell theory the constitutive equations for transverse stresses are not satisfied pointwise [13] but may be fulfilled in an integral sense. In particular, for the transverse normal component the following integral equation [13] should be adopted

$$\sum_k \int_{\delta_{k-1}}^{\delta_k} \left(\sigma_{33}^{(k)} - \sum_{\gamma,\delta} C_{33\gamma\delta}^{(k)} \varepsilon_{\gamma\delta} - C_{3333}^{(k)} \varepsilon_{33} \right) d\alpha_3 = 0. \tag{18}$$

Taking into account relations (2) and introducing notations

$$H_{33}^{(k)} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{33}^{(k)} d\alpha_3, \qquad n_k^{\pm} = \int_{\delta_{k-1}}^{\delta_k} N^{\pm}(\alpha_3) d\alpha_3, \tag{19}$$

one can rewrite equation (18) as follows

$$\sum_k \left[H_{33}^{(k)} - \sum_{\gamma,\delta} C_{33\gamma\delta}^{(k)} \left(n_k^{-} e_{\gamma\delta}^{-} + n_k^{+} e_{\gamma\delta}^{+} \right) - h_k C_{3333}^{(k)} e_{33} \right] = 0. \tag{20}$$

This allows us to assume that the transverse normal stress is independent on the thickness coordinate α_3 that may be appreciated as a good remedy [10, 11] for overcoming the thickness locking phenomenon. So, following this idea, the transverse normal stress is assumed to be constant in the thickness direction for every layer of the shell

$$\sigma_{33}^{(k)} = \frac{1}{h_k} H_{33}^{(k)} = \frac{1}{h_k} \sum_{\gamma,\delta} C_{33\gamma\delta}^{(k)} \left(n_k^{-} e_{\gamma\delta}^{-} + n_k^{+} e_{\gamma\delta}^{+} \right) + C_{3333}^{(k)} e_{33}. \tag{21}$$

Substituting further thickness stress (21) in Eq. (10) and integrating these modified equations together with remaining constitutive equations (16) and (21) across the shell thickness with account for relations (2), (6), (7) and (13), one derives

$$\mathbf{H} = \mathbf{D}\mathbf{e}. \tag{22}$$

Here, **D** denotes a constitutive stiffness matrix introduced in Section 4 whose components are defined as

$$D^{pq}_{\alpha\beta\gamma\delta} = \sum_k \left(n^{pq}_k Q^{(k)}_{\alpha\beta\gamma\delta} + \frac{n^p_k n^q_k \mu^{(k)}_{\alpha\beta 33} \mu^{(k)}_{33\gamma\delta}}{h_k \Lambda_k} \right),$$

$$D^{\pm}_{\alpha\beta 33} = -\sum_k \frac{1}{\Lambda_k} n^{\pm}_k \mu^{(k)}_{\alpha\beta 33}, \qquad D^{\pm}_{33\alpha\beta} = -\sum_k \frac{1}{\Lambda_k} n^{\pm}_k \mu^{(k)}_{33\alpha\beta},$$

$$D^{pq}_{\alpha 3\beta 3} = \sum_k n^{pq}_k C^{(k)}_{\alpha 3\beta 3}, \qquad D_{3333} = \sum_k \frac{1}{\Lambda_k} h_k,$$

$$n^{pq}_k = \int_{\delta_{k-1}}^{\delta_k} \left[N^-(\alpha_3) \right]^{2-p-q} \left[N^+(\alpha_3) \right]^{p+q} d\alpha_3,$$

where instead of (19) the more suitable notations $n^0_k = n^-_k$ and $n^1_k = n^+_k$ have been used, and throughout this section superscripts p, q take the values 0 and 1.

It should be mentioned that this *modified* constitutive law is quite efficient for most engineering problems. But for incompressible or nearly incompressible materials [9] some difficulties leading to volumetric locking can occur.

5.3 Reduced constitutive equations

In order to circumvent simultaneously thickness and volumetric locking, we invoke a standard engineering assumption $\sigma^{(k)}_{33} << \sigma^{(k)}_{\alpha\beta}$. This implies that coefficients $A^{(k)}_{\alpha\beta 33}$ should be formally set to zero in equations of Hooke's law (9) for the in-plane strains. At the same time the last equation for the thickness strain is left unchanged, i.e., $A^{(k)}_{33\alpha\beta} \neq 0$. Allowing for this assumption and using a technique described in Section 5.1, one arrives at constitutive equations (10), (12) and (16), where

$$\mu^{(k)}_{\alpha\beta 33} = 0 \quad \text{and} \quad \Lambda_k = A^{(k)}_{3333}$$

in accordance with relations (11) and (13).

Integrating these reduced constitutive equations across the shell thickness (see Section 5.2), the following expressions for components of the constitutive stiffness matrix **D** are obtained:

$$D^{pq}_{\alpha\beta\gamma\delta} = \sum_k n^{pq}_k Q^{(k)}_{\alpha\beta\gamma\delta}, \qquad D^{\pm}_{\alpha\beta 33} = 0, \qquad D^{\pm}_{33\alpha\beta} = -\sum_k \frac{1}{A^{(k)}_{3333}} n^{\pm}_k \mu^{(k)}_{33\alpha\beta},$$

$$D^{pq}_{\alpha 3\beta 3} = \sum_k n^{pq}_k C^{(k)}_{\alpha 3\beta 3}, \qquad D_{3333} = \sum_k \frac{1}{A^{(k)}_{3333}} h_k.$$

The *reduced* constitutive law was proposed in works [13, 14] for overcoming thickness locking and showed a good performance in case of using the Timoshenko-Mindlin shell theory. It should be mentioned that this approach yields the non-symmetric material matrix and, as a result, more computational efforts have to be made.

5.4 Simplified constitutive equations

When a shell is undergone pure bending, one half of the shell body in the thickness direction is under tension and the other half is under compression, i.e., the thickness strain according to the complete 3D Hooke's law would be zero due to the limitation of the linear displacement approximation (1). So, a shell will be in the plane strain state instead of the expected plane stress state. In order to circumvent these difficulties, the simplified constitutive stiffness matrix can be employed

$$D^{pq}_{\alpha\beta\gamma\delta} = \sum_k n^{pq}_k Q^{(k)}_{\alpha\beta\gamma\delta}, \qquad D^{\pm}_{\alpha\beta 33} = D^{\pm}_{33\alpha\beta} = 0,$$

$$D^{pq}_{\alpha 3\beta 3} = \sum_k n^{pq}_k C^{(k)}_{\alpha 3\beta 3}, \qquad D_{3333} = \sum_k \frac{1}{A^{(k)}_{3333}} h_k .$$

This is due to the plane stress enforcement which is done by decoupling the transverse normal stress with all other stresses in the 3D Hooke's law [5, 7, 8, 12], i.e., it is supposed that

$$A^{(k)}_{\alpha\beta 33} = A^{(k)}_{33\alpha\beta} = 0.$$

It is apparent that the *simplified* constitutive law leads to the symmetric constitutive stiffness matrix **D** but it is slightly deficient for the thick anisotropic shells. So, allowing for a simplicity of such approach it may be recommended for an analysis of composite thin-walled structures.

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Об использовании 6-параметрических моделей многослойных оболочек в механике конструкций

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Ключевые слова и фразы: анизотропия; заклинивание по толщине; теория многослойных оболочек первого порядка.

Аннотация: Рассмотрены новые геометрически точные модели многослойных оболочек. Эти модели основаны на объективных деформационных соотношениях, представленных в локальных криволинейных координатах, и поэтому могут быть использованы для построения эффективных криволинейных элементов многослойных оболочек. Однако практическое использование таких элементов требует развития соотношений упругости, для того чтобы преодолеть Пуассоновское и объемное заклинивания. С этой целью изучаются и сравниваются три типа модифицированной материальной матрицы жесткости.

Über die Benutzung der 6-parametrischen Modelle der vielschichtigen Mäntel in der Konstruktionsmechanik

Zusammenfassung: Es sind neue geometrisch genaue Modelle der vielschichtigen Mäntel untersucht. Diese Modelle sind auf die objektiven in lokalen krummlinigen Koordinaten vorgestellten Deformationsverhältnisse gegründet. Deshalb können sie für die Konstruktion der effektiven krummlinigen Elemente der vielschichtigen Mäntel verwendet werden. Doch fordert die praktische Anwendung solcher Elemente die Entwicklung der Elastizitätsverhältnisse, um Puasson- und Raumfestklemmen zu überwinden. Zu diesem Zweck werden 3 Arten der abgeänderten materiellen Matrix der Steifheit studiert und verglichen

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Sur l'utilisation des modèles multicouches à 6-paramètres pour les enveloppes dans la mécanique des constructions

Résumé: Sont envisagés de nouveaux modèles multicouches pour des enveloppes qui sont présis géométriquement. Ces modèles sont fondés sur les relations déformationnelles présentées dans les coordonnés locales cuvilignes, c'est pourquoi ils peuvent être utilisés pour la construction des éléments efficaces des enveloppes multicouches. Toutefois l'emploi pratique de tels éléments demande un développement du rapport de l'élasticité pour surmonter le coinçage Poisson et celui de volume. Dans ce but sont étudiés et comparés trois types de la matrice matérielle modifiée de la rigidité.
