CONSTRUCTION OF SOLUTIONS OF ONE CLASS AUTONOMOUS SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS*

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Recommended for Publication by Professor G.M. Kulikov

Abstract: Based on the method of successive approximations of Picard there constructed a solution of autonomous systems of ordinary differential equations which right-hand side is a multivariate polynomial. There given the results of numerical experiment that may allow a new look at the well-known hypothesis about the structure of the attractor of the Lorenz system.

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1. Introduction. Letэs consider the normal autonomous system of ordinary differential equations which vector notation has the following form

$$\dot{x} = Ax + b(x), \tag{1}$$

where $x = (x^1, ..., x^n)$ – vector function of a real variable t; $A = (a^i_j)$ – real $(n \times n)$ – matrix and $b = (b^1, ..., b^n)$ – real vector function, which every element b^i is a polynomial of variables $x^1, ..., x^n$, in this case the degree of the polynomial b^i and b^j may not match.

Systems of the form (1) for a long time represent sufficiently much interest for applications because many process models of different physical nature are described by similar systems [1–3]. Particular interest the system of the form (1) recently purchased since all known system that, supposed to contain strange attractors were precisely those systems [3].

To obtain the solutions of the system (1) it is generally used a standard methods of numerical analysis that do not take into account the specific form of the right-hand side of the system. The last obviously leads to the fact that any of the used methods can not be considered as optimal. The aim of this work is to develop method of construction of

solutions of the system (1), which takes into account the fact that each element of the function b is a polynomial of variables $x^1, ..., x^n$.

2. Local solutions of the system (1). To construct a local solution $x(t)$ of system (1) that satisfies the initial condition

$$x(0) = x_0, \tag{2}$$

let's replace (1) by integral equation

$$x(t) = x_0 + \int\limits_0^t \left[Ax(\tau) + b(x(\tau)) \right] d\tau. \tag{3}$$

Let's denote α and q as some positive numbers, and Γ – compact part $(n+1)$ – dimensional Euclidean vector space $\mathbb{R}^{n+1}$ that denied by the inequations

$$|x - x_0| \le \alpha, \quad |t| \le q .$$

Suppose also that $r \le q$ – positive number that will be denied below. Together with Γ let's introduce more "narrow" compact set $\Gamma_r \subset \mathbb{R}^{n+1}$ that defined by the inequations

$$|x - x_0| \le \alpha, \ |t| \le r. \tag{4}$$

Further let's denote Π_r – set of continuous functions which graphics are contained in Γ_r. Let's consider the operator

$$A\varphi = x_0 + \int\limits_0^t \left[A\varphi(\tau) + b(\varphi(\tau)) \right] d\tau. \tag{5}$$

Since the set

$$\Sigma = \left\{ x \in \mathbb{R}^n : |x - x_0 \le \alpha| \right\} \tag{6}$$

is compact there can be found such positive number M, that for all $x \in \Sigma$ the following inequation is satisfied

$$|Ax + b(x)| \le M. \tag{7}$$

Then from the condition

$$r \le \frac{\alpha}{M} \tag{8}$$

it will follow that the operator (5) is an operator that displays set Π_r into itself [4]. Therefore let's assume that the number r in (4) is chosen such that inequation (8) is satisfied. Now let's notice that the system (1) has a unique solution $x(t)$ with the initial condition (2), defined on some interval $[-T, T]$. Therefore, the equation (3) also has a unique solution $x(t)$, defined on the interval $[-T, T]$. To find this solution let's use Picard's method of successive approximations and write

$$x_{N+1}(t) = x_0 + \int\limits_0^t \left[Ax_N(\tau) + b(x_N(\tau)) \right] d\tau. \tag{9}$$

Acting as usual let's put

$$x_1(t) \equiv x_0. \tag{10}$$

Then it is easy to notice that there exists some iteration

$$A_p\varphi = \underbrace{A...A}_{p}\varphi$$

of operator A that is the compression [4]. Therefore, method of successive approximations (9) that satisfies the condition (10) on the interval $[-r, r]$ is uniformly converges to the solution $x(t)$. It remains to construct the last. To find $x(t)$, first of all let's note that due to (9) and (10) for all values $t \in [-r, r]$ the following equation is valid

$$x_2(t) = x_0 + \int_0^t [Ax_0 + b(x_0)]d\tau = x_0 + Atx_0 + b(x_0)t. \tag{11}$$

Now, substituting (11) into (9) when $N = 2$ and $t \in [-r, r]$ we can write

$$x_3(t) = x_0 + \int_0^t [A(x_0 + A\tau x_0 + b(x_0)\tau) + b(x_0 + A\tau x_0 + b(x_0)\tau)]d\tau =$$

$$= x_0 + Atx_0 + \frac{(At)^2}{2!}x_0 + Ab(x_0)\frac{t^2}{2!} + \sum_{k=1}^{\theta_2} f_{2,k}(x_0)t^k,$$

where θ_2 – some positive number depends on the type of form $b(x_0)$; $f_{2,k}$ – corresponding real vector functions that defined and continuous on a set Σ such that

$$\sum_{k=1}^{\theta_2} f_{2,k}(x_0)t^k = \int_0^t b(x_0 + A\tau x_0 + b(x_0)\tau)d\tau.$$

If it takes the

$$\sum_{k=1}^{\theta_2} f_{N,k}(x_0)t^k = A^{N-2}\sum_{i=1}^{\theta_2} f_{2,i}(x_0)t^{i+N-2}\frac{i!}{(i+N-2)!} +$$

$$+ A^{N-3}\sum_{i=1}^{\theta_3} f_{3,i}(x_0)t^{i+N-3}\frac{i!}{(i+N-3)!} + \ldots + A\sum_{i=1}^{\theta_{N-1}} f_{N-1,i}(x_0)\frac{t^{i+1}}{i+1} +$$

$$+ \int_0^t b(x_N(\tau))d\tau, \tag{12}$$

then it is easy to verify by induction that in the general case at $t \in [-r, r]$ we have

$$x_{N+1}(t) = x_0 + Atx_0 + \frac{(At)^2}{2!}x_0 + \ldots + \frac{(At)^N}{N!}x_0 +$$

$$+ A^{N-1}b(x_0)\frac{t^N}{N!} + \sum_{k=1}^{\theta_N} f_{N,k}(x_0)t^k, \tag{13}$$

where θ_N is some positive number as θ_2 depends solely on the type of form $b(x_0)$, and $f_{N,k}$ – corresponding real vector functions that obviously defined and continuous on a set Σ.

R e m a r k   1. If the form $b(x_0)$ is non-linear then how easy to verify by induction

$$\lim_{N \to \infty} \frac{N}{\theta_N} = 0$$

and moreover

$$\lim_{N \to \infty} (\theta_{N+1} - \theta_N) = \infty. \tag{14}$$

It is obvious that due to the condition (8) the method of successive approximations (9) converges to the solution $x(t)$ of equation (3) uniformly on the interval $[-r, r]$.

That is why passing in equation (9) to the limit at $N \to \infty$, considering (13) we obtain the representation

$$x(t) = \exp(At)x_0 + f(x_0, t), \tag{15}$$

valid for all value $t \in [-r, r]$, where at a fixed x_0 function

$$f(x_0, t) = \lim_{N \to \infty} \sum_{k=1}^{\theta_N} f_{N,k}(x_0) t^k \tag{16}$$

is defined and continuous on the interval $[-r, r]$ moreover convergence in the equation (16) is uniform on $[-r, r]$. But the choice of the initial condition (2) above essentially played no role. Therefore, it is easy to see the validity of the following.

T h e o r e m 1. Assume that the number $\alpha > 0$ set also for its corresponding number $M > 0$ on a set Σ inequation (7) is satisfied. Then for each point $x_0 \in \Sigma$ there exists a positive number

$$T = \frac{\alpha}{M}, \tag{17}$$

that on each of the intervals $[r_1, r_2] \subset [-T, T]$ the solution $x(t)$ of system (1) with the initial condition (2) can be obtained by uniformly convergent method of successive approximations (9). At that on the interval $x(t)$ the solution $x(t)$ satisfies the equation (15).

3. Non-locally continuous restricted solutions. Now let's $x(t)$ some solution of the system (1) with the initial condition (2), defined for all values $t \geq 0$ and contained at these values t in a set Σ, given by the equation of the form (6) in which α – some suitably chosen positive number. Then there can be found a sufficiently large positive number M, that at $x \in \Sigma$ inequation (7) is satisfied.

For define above numbers α and M let's denote the number of T by the formula (17). Then according to Theorem 1 the solution $x(t)$ satisfies the equation (15) on the interval $[0, T]$. At all values $t \geq 0$ let's put

$$x_K(t) = x(t + (K-1)T), \quad K = 1, 2, \dots \tag{18}$$

It is easy to see that each function of the family (18) is a solution of the system (1), defined for all values $t \geq 0$ and contained at these values t in set Σ. On the other hand, due to the limit of solution $x(t)$ the number M could initially be chosen so that when the conditions

$$|x - x_k(t)| \leq \alpha, \quad |t| \leq T$$

are satisfied for all values $K = 1, 2, \dots$ it was also is satisfied the inequation (7). That is why if we take

$$x_0(T) = x_0,$$

then due to Theorem 1 at $t \in [0, T]$ we have

$$x_K(t) = \exp(At)x_{K-1}(T) + f(x_{K-1}(T), t), \quad K = 1, 2, \dots,$$

where at a certain $x_{K-1}(T)$ function

$$f(x_{K-1}(T), t) = \lim_{N \to \infty} \sum_{k=1}^{\theta_N} f_{N,k}(x_{K-1}(T)) t^k \tag{19}$$

is defined and continuous on the interval $[-T, T]$, moreover convergence in the equation (19) is uniform on $[-T, T]$. Then, as a trivial consequence of Theorem 1 it is valid the following very important to build solutions of system (1).

T h e o r e m   2. For all values $t \in [0, T]$ the defined above solution $x(t)$ satisfies the condition

$$x(t + (K-1)T) = \exp(At)x((K-1)T) + f(x((K-1)T), t), \qquad (20)$$

in which $K = 1, 2, ...$

R e m a r k   2. If $x(t)$ solution of the system (1) with the initial condition (2) defined for all values $t \le 0$ and contained at these values t in a set Σ, then at a certain above number T on the interval $[-T, 0]$ an equation

$$x(t + (1-K)T) = \exp(At)x((1-K)T) + f(x((1-K)T), t), \qquad (21)$$

is valid in which $K = 1, 2, ...$ Thus if the solution $x(t)$ is defined for all values $|t| < \infty$ and contains at these values t in a set Σ then, depending on the value of t it satisfies one of the equations (20) or (21).

R e m a r k   3. Generally speaking explicit selection of the linear part in the system (1) outlines a slightly different way of constructing non-locally continuous solutions of given system. This method is based on the fact that, by Theorem 2 for all values $t \in [0, T]$ and $K = 1, 2, ...$ each solution $x(t)$ of system (1), defined and limited at $t \ge 0$ satisfies the inequation

$$x(t + (K-1)T) = \exp(At)\sum_{i=1}^{K} \exp(A(K-i)T)f(x((i-2)T), T) +$$
$$+ f(x((K-1)T), t), \qquad (22)$$

where

$$f(x(-T), T) = x_0.$$

The ratio (22) is very promising for topological research of solutions of the system (1). In addition, it can be obviously used also for high-precision computational experiments.

4. Approximate construction of solutions. Again, we consider the solution $x(t)$ of system (1) with the initial condition (2), defined for all values $t \ge 0$ and contained at these values t in a set Σ and fix the number T that satisfying the conditions of Theorem 1 and Theorem 2. Let's denote

$$x(0), x(T), ..., x(KT), ... \qquad (23)$$

position of the system (1) at moments of time

$$0, T, ..., KT, ... \qquad (24)$$

Further using the equality

$$g^t x_0 = \exp(At)x_0 + f(x_0, t). \qquad (25)$$

Let's consider the operator g^t. Then according to Theorem 1 at $t \in [0, T]$ we have

$$x(t) = g^t x_0.$$

In particular

$$x(T) = g^T x_0,$$

and hence, due to Theorem 2

$$x(KT) = \underbrace{g^T ... g^T}_{K} x_0, \quad K = 0, 1, ...$$

Thus if the operator g^t is built the ratio (26) uniquely defines the positions (23) of system (1) at the moments of time (24). By virtue of the relations (12)−(16) the exact construction of the operator g^t is very difficult if at all possible even in the simplest cases. Therefore let's henceforth restrict by the approximate construction g^t. For any natural number N let's put

$$g_N^t x_0 = x_0 + Atx_0 + \frac{(At)^2}{2!} x_0 + ... + \frac{(At)^N}{N!} x_0 +$$

$$+ A^{N-1} b(x_0) \frac{t^N}{N!} + \sum_{k=1}^{\theta_N} f_{N,k}(x_0) t^k,$$

thereby determining the approximation g_N^t to the operator g^t. Then due to the equation (16) for every positive number ε there can be found such natural number N_ε, that at $N > N_\varepsilon$ here satisfies an inequation

$$\max_{0 \le t \le T} \left| g_N^t x_0 - g^t x_0 \right| < \varepsilon.$$

Moreover, by construction, for each fixed $K = 1, 2, ...$ at given ε the number $N > N_g$ can be chosen such that

$$\left| g_N^T x(KT) - g^T x(KT) \right| < \varepsilon.$$

In other words, with any given beforehand accuracy ε an operator g_N^T can be constructed such that at fixed value T there valid an inequations

$$\left| g_N^T x(KT) - x((K+1)T) \right| < \varepsilon, \quad K = 0, ..., L, \tag{26}$$

where L a sufficiently large positive number that characterizes the time $(L+1)T$ of the system (1) functioning finite for concrete calculations.

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Построение решений одного класса автономных систем обыкновенных дифференциальных уравнений

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Ключевые слова и фразы: автономные системы; аттрактор; обыкновенные дифференциальные уравнения; последовательные приближения; система Лоренца.

Аннотация: На основе метода последовательных приближений Пикара строится решение автономных систем обыкновенных дифференциальных уравнений, правая часть которых представляет собой многомерный многочлен. Приводятся результаты численного эксперимента, возможно позволяющие по новому взглянуть на известную гипотезу о структуре аттрактора системы Лоренца.

Konstruktion der Lösungen einer Klasse der autonomen Systeme der gewöhnlichen Differentialgleichungen

Zusammenfassung: Aufgrund der Methode der konsequenten Approximationen von Picard wird die Lösung der autonomen Systeme der gewöhnlichen Differentialgleichungen, deren rechter Teil das multidimensionale Polynom darstellt, gebaut. Es werden die Ergebnisse des numerischen Experimentes, die möglich neu auf die bekannte Hypothese über die Struktur des Attraktors des Lorenz-Systems anschauen zulassen, angeführt.

Construction des solutions d'une classe des systèmes autonomes des équations différentielles ordinaires

Résumé: A la base de la méthode des approximations successives de Picard est construite la solution des systèmes autonomes des équations différentielles ordinaires dont la première partie représente le polynôme multidemensionnel. Sont cités les résultats de l'expériment numérique qui permettent probablement d'observer de la nouvelle manière l'hypothèse connue sur la structure de l'attracteur du système de Lorenz.

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