УДК 517.958:52/59

MODELING OF PROCESSES OF MECHANICAL VIBRATIONS IN A COURSE OF MATHEMATICS

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Key words and phrases: Fourier method; mathematical models; professional competencies; technological ways of teaching.

Abstract: In the context of competence approach, technological ways of teaching mathematical modeling in engineering high schools are offered; these include an algorithm for construction, research, and interpretation of mathematical models and their application to problems of mechanical vibrations.

In the transition to a two-tier system of higher education the problem of convergence of "theoretical" and "real" mathematics by means of the effective use of the ideas and methods of mathematical modeling is particularly relevant. The ability to mathematical modeling is a "pass-through" competence, i.e. it is common for both bachelor and master and differs only in the level of mastering. For example, in accordance with the requirements of the Federal State Educational Standard of Higher Professional Education, Bachelor of Bachelor degree 270800 "Building" should be able "to use the basic laws of the natural sciences in professional activities, the application of methods of mathematical analysis and modeling, theoretical and experimental research" (professional competence PC-1). In turn, the Master of the direction of Master 270800 need to be able to "develop the physical and mathematical models of phenomena and objects related to the profile of the activity (PC-19), the person should be prepared to address a number of professional tasks in accordance with the basic educational program and forms of professional activities, which include: mathematical modeling of processes in structures, systems, computer-based implementation of models, the development of computational methods and computer-aided design". In the study of the mathematics student should, in particular, know:

– the main approaches to the formalization and modeling of various processes (e. g., motion and equilibrium of material bodies and mechanical systems);

be able to:

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– independently use mathematical tools contained in the literature;

– formulate the physical and mathematical formulation of research problems;

own:

– the primary skills and basic techniques of solving mathematical problems of general engineering and specialized disciplines profiling;

– mathematical tools for the development of mathematical models of processes and phenomena and solving practical problems of professional activity [3].

1. The aim of this work is to develop some technological methods of instruction in mathematical modeling of engineering schools and the implementation of these techniques on the example of modeling of mechanical vibrations [4]. In particular, the basic conceptual series, algorithm of constructing, studying and interpretation of mathematical models and its application to a specific problem are proposed.

Getting to the solution of problems by means of mathematical modeling, student should possess the following concepts and facts.

Modeling as a replacement of an object A (the original) by another object M (model) in many cases allows a researcher to investigate interesting properties of the original, modeling A on the main characteristics. *The purpose of modeling* is receiving, processing, presentation and use of information about the objects that interact with each other and the outside world, so the model is a mean of knowledge of the behavior and properties of the object.

The problems solved by the models are based on the following *general scheme of presentation of model*

$$X \to W \to Y.$$

Here X is an *input vector (exogenous) variable*; Y is a *vector of output variable* (model outcomes); W is the so-called *operator of the model* (the operator of the model converts the input information into output in accordance with the problem which is solved on the model). The following three variants of mentioned problems are possible:

1) *direct problem*: X and W are given, to find Y;

2) *the inverse problem* 1: Y and W are given, to find X;

3) *the inverse problem* 2: X and Y are given, to find W.

In the latter problem the cases of “black box” when the operator of the system is completely unknown and the cases of “gray box” when the structure of the operator is given and the values of parameters are not given are possible.

Approximate representation of real objects, processes, or systems, expressed in mathematical terms and preserved the essential features of the original is called a *mathematical modeling*. Properties of the object (process, system), its parameters, internal and external relations in a mathematical model are described in a quantitative form with the help of logical-mathematical structures. In other words, the application task is “translated” into the formal language of mathematics (the constructing of the model), and can be solved by means of the same mathematics. The obtained solution is applied to the object of studying, i.e. the consequences derived from the model in the language of mathematics are interpreted into the language adopted in a given subject area [2].

2. The main technological method of teaching mathematical modeling is the implementation of the next *step-by-step algorithm*.

Step 1. Consideration of the so-called informative model.

1.1. The main properties and relations of the real object are formulated in terms of the original subject area.

1.2. The postulates of the model, i.e. the basic hypothesis supported, as a rule, empirically (for example, the hypothesis of the linear nature of the studied dependence, etc.), are formulated.

1.3. A valid within this situation simplification (distraction from different kinds of «faultiness», neglect of the action of certain forces, etc.) is made.

Step 2. Formalization of the model, i.e. transition to the description of the model in mathematical terms.

2.1. Introduction of basic studied mathematical objects (variables, functions).

2.2. Formalization of relations between objects that are written in the form of equations or inequalities in general case.

2.3. Statement of the corresponding mathematical problem.

Step 3. Studying of the received mathematical model.

3.1. Simplification of the mathematical model (some assumptions: for example, the replacement of finite differences by relevant differentials, the use of asymptotic formulas), in the case of alleviation of solution of the problem.

3.2. Choice of the method of solution; in particular, it is possible using of a method of "internal mathematical" modeling, i.e. transfer of problems to the language of the "adjacent" mathematical disciplines (e.g., the use of vector-coordinate method in the solution of stereometric problems, replacing of the functions by its decomposition into a series of orthogonal system, etc.).

3.3. Receipt of the answer in the mathematical problem. Informative component.

Step 4. The interpretation of the model, i.e. extraction (on the basis of realized mathematical model) information about the studied object or a process in terms of the original subject area.

4.1. Finding the numerical values of parameters of the process.

4.2. Construction of the corresponding diagrams, graphs etc.

4.3. Interpolation / extrapolation of the process.

According to the nature of the solved problems the implementation of some of the prescriptions contained in sub-paragraphs of this algorithm may be absent or their order may be changed.

Interpreting model, it is necessary to consider the requirement of *adequacy* of the received solution of a mathematical problem to real process or the phenomenon. For example, if the oscillation of the string in its assumption of infinite length is investigated, its real size should be considered returning to the real object; if the oscillations are damped in time infinitely, they practically stopped after a sufficiently great length of time in the real process, etc.

In connection with the stage of *interpretation*, the conception of *transitivity* model is introduced. The essence of it is that if some processes or phenomena have a common mathematical model, then one of them can be used as a model for the other.

3. The process of mathematical modeling of mechanical vibrations and heat and mass transfer processes, in general terms, consists of the following. A set of physical properties of objects subjected to mechanical vibrations (string, thread) or thermophysical characteristics of materials where heat exchange takes place is chosen as a vector of input variables. The operator W of the model (a differential equation in partial derivatives with boundary and initial conditions imposed on its decision) is not specified in advance: building of W is based on relevant physical laws and it is a part of the process of mathematical modeling. The corresponding "outputs of model" are determined on the basis of the solution of the obtained boundary value problem by the method of Fourier [1].

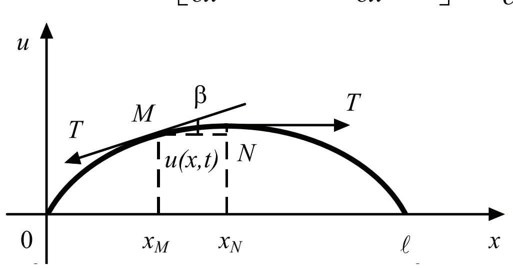
4. Let us consider the applying of specified in paragraph 2 algorithm for the problem of modeling of the process of free small transverse vibrations of the string and accompany the consideration with relevant methodical recommendations.

Step 1. Vector of input variables is defined by the parameters of the string. Wherein, a string is a flexible elastic thread, and it means that the tensions, which appear in the string at any period of time, are directed tangentially to its instantaneous profile, i. e. the string does not resist bending. Mentioned parameters of the string are the length of the string l, density of the material ρ (the string is considered to be homogeneous), the area of transverse section S and the force of the string tension T. The sought-for output of mathematical model is a function $u(x,t)$ (*step 2.1*) which describes the vertical displacement of the point with abscissa x at the moment of time t (figure) of the string with the endings rigidly fixed at points $x = 0$ and $x = l$ (external forces absent, *step 1.3*).

The following assumptions are possible due to the smallness of vibrations (*step 3.1*): $\frac{\partial u}{\partial x} = \mathrm{tg}\beta \approx \sin\beta$ and $\cos\beta \approx 1$, where β is an angle between the tangent to the curve $u(x, t)$ and the X-axis (*step 3.1*). By virtue of these assumptions the tension of the string does not depend on variables x and t, i.e. $T(x_M, t) = T(x_N, t) = T$.

Step 2. The tension forces T at the endings of a small element of the string MN are tangential to the string. Let coordinate points be $x_M = x$ and $x_N = x + dx$, and the angles between tangents and X-axis are β and $\beta + d\beta$. Considering vertical displacements of the points projection forces on the U-axis should be determined as

$$T\sin(\beta + d\beta) - T\sin\beta \approx T\left[\frac{\partial u}{\partial x}(x + dx,t) - \frac{\partial u}{\partial x}(x,t)\right] = T\frac{\partial^2 u}{\partial x^2}(x,t)dx.$$



Vertical displacement of the points of the string

The weight of the element *MN* is equal to $dm = \rho(x)Sdx,$ and then an equation of motion of the element *MN* can be written as (*step 2.2*)

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} dx$$

or

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $a = \sqrt{\dfrac{T}{\rho S}}$. The sought-for function $u(x, t)$ will satisfy the boundary conditions

$$u(0,t) = 0, \;\; u(\ell,t) = 0, \tag{2}$$

in the case when the string is rigidly fixed at the ends.

Since the process of transverse vibrations of the string depends on its initial form and the velocity distribution at the initial period of time $t = 0$, the initial conditions must be specified:

$$u(x,0) = \varphi(x), \;\; \frac{\partial u}{\partial t}(x,0) = \psi(x), \tag{3}$$

where $\varphi(x)$, $\psi(x)$ are the given functions of x coordinate. Thus (*step 2.3*), a boundary value problem is set (1) – (3).

To solve this problem a student should possess the following skills:

– to find a fundamental system and build a general solution of the homogeneous linear differential equation of second order with constant coefficients;

– to represent the function as the sum of trigonometric series and find its Fourier coefficients.

Step 3. The solution of this problem can be found using the method of separation of variables (Fourier method), the essence of which is to represent not identically equal to zero and satisfying the boundary conditions (2) solutions in the form of the product

$$u(x,t) = X(x)T(t),$$

where $X(x)$ is a function that depends only on the abscissa x, $T(t)$ is a function that depends only on the time t. The intermediate result of the method (*step 3.2*) is a series of relations

$$T''X = a^2 TX''$$

or

$$\frac{T''}{a^2 T} = \frac{X''}{X},$$

the latter of which is only possible if its left side does not depend on t, and the right side does not depend on x, i.e. both sides are constant

$$\frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda = \text{const.} \tag{4}$$

Wherein the boundary conditions (2) are transformed to

$$X(0) = 0, X(\ell) = 0, \tag{5}$$

where $\lambda > 0$ is a constant; in the step 3.2 the fact that the case $\lambda \le 0$ is not compatible with the conditions (5) is established; it is recommended that the student should come to the specified intermediate result through self-study.

The problem (4) – (5) can be considered as an operator of the mathematical model, and its further reduction leads to a system of two ordinary differential equations

$$X'' + \lambda X = 0, \tag{6}$$

$$T'' + a^2 \lambda T = 0. \tag{7}$$

Further actions in the steps 3.2, 3.3 are an obtaining of a fundamental system of solutions (6) and its general solution in the form of

$$X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x. \tag{8}$$

Reasoning for $T(t)$ is analogous; the general solution is in form of

$$T(t) = A\cos a\sqrt{\lambda}t + B\sin a\sqrt{\lambda}t. \tag{9}$$

The next stage is the choice of parameters A, B, C, D and λ which are generated by the actions of the model's operator. The substitution of values $x = 0$ and $x = \ell$ in (8) and conditions (4.5) leads to relations

$$X(0) = C = 0, X(\ell) = D\sin\sqrt{\lambda}\ell = 0.$$

Since the function $X(x)$ is not identically equal to zero, then $D \ne 0$, and therefore $\sin\sqrt{\lambda}\ell = 0$, from which

$$\lambda = \lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, ... \tag{10}$$

Thus, the non-trivial solutions of the boundary-value problem are only possible in the case of (10). Values $\lambda = \lambda_n$ are corresponded to the functions such as

$$X_n(x) = D_n \sin\frac{n\pi}{\ell}x, \quad n = 1, 2, ...,$$

which are defined up to the arbitrary factors D_n. The same values of λ_n correspond to solutions of the equation (4.7) which were presented in the form (9):

$$T_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t,$$

where A_n, B_n are arbitrary constants and $\omega_n = \frac{n\pi a}{\ell} = \frac{n\pi}{\ell}\sqrt{\frac{T}{\rho_1}}$, $n = 1, 2, ...$

Private solutions of the equation (1) satisfying the boundary conditions (2) now take the form

$$u_n(x,t) = (a_n \cos\omega_n t + b_n \sin\omega_n t)\sin\frac{n\pi}{\ell}x, \tag{11}$$

where $a_n = A_n D_n$, $b_n = A_n D_n$.

The last stage of constracting of the mathematical model's output is to implement the following idea which is fundamental for the Fourier method. Since any of functions of the form (11) for $n = 1, 2, ...$ as well as any their finite sum generally speaking does not satisfy the initial conditions (3) for sufficiently random assignment of functions $\varphi(x)$ and $\psi(x)$, then the solution of the boundary value problem should be sought in the form of *an infinite sum* (*step 3.2* which is called "internal mathematical" modeling)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin \frac{n\pi}{\ell} x. \quad (12)$$

Now the coefficients of the trigonometric series a_n and b_n can be found as the Fourier coefficients of expansions

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{\ell} x$$

and

$$\psi(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \omega_n b_n \sin \frac{n\pi}{\ell} x.$$

It follows,

$$a_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \sin \frac{n\pi}{\ell} x dx \quad \text{and} \quad b_n = \frac{2}{\ell \omega_n} \int_0^{\ell} \psi(x) \sin \frac{n\pi}{\ell} x dx. \quad (13)$$

Thus (*step 3.3*), the output of the mathematical model of the process of mechanical vibrations of the string is obtained; it is the sum of series (12) with coefficients (13).

The specificity of this mathematical problem is the necessity to establish convergence of the series (12) and the possibility of its term differentiation; it was proved only that its general term is the solution of the equation (1), and the question about preservation of this property for the infinite sum was opened. The reasoning here is as follows (component of justification 3.3). Convergence and term differentiable (twice on x, twice on t) of the series (12) take place if this series and the series from corresponding derivatives are majorized. As it turns out, majorant series are the series of the form

$$\sum_{n=1}^{\infty} n^2 |a_n| \quad \text{and} \quad \sum_{n=1}^{\infty} n |b_n|.$$

Their convergence will take place if:

a) Derivatives $\varphi'(x)$ and $\varphi''(x)$ are continuous, $\varphi'''(x)$ is piecewise continuous and $\varphi(0) = \varphi(\ell) = 0$; $\varphi''(0) = \varphi''(\ell) = 0$;

b) $\psi'(x)$ is continuous, $\psi''(x)$ is piecewise continuous and $\psi(0) = \psi(\ell) = 0$.

5. Let us consider an interpretation of the model (step 4) on a concrete example. Let the velocity $v = 5$ m/s be reported to the points of rectilinear uniform string of length $\ell = 0.5$ m having a rectangular cross-sectional size 0,25×0,4 mm at the initial period of time $t = 0$. The string tension $T = 40$ N, the bulk density of the material of the string $\rho = 6.4 \cdot 10^3$ kg/m^3. Find the first five natural frequencies and the shape of the string at time t, if its ends $x = 0$ and $x = \ell$ are rigidly fixed. The resistance of the medium is neglected.

The central point in the analysis of mathematical models of the natural vibration of the string is to obtain a solution of the corresponding boundary value problem in the form of (see (12) – (13))

$$u(x,t)=\frac{1}{25\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\sin\omega_{2k-1}t\sin\frac{\omega_{2k-1}}{250}x,$$

where $\omega_{2k-1}=500(2k-1)\pi.$

An example of this type can be related to the so-called case-tasks (the level of “know”, “able”, “possess”) representing a key component of pedagogical measuring materials. An implement of such tasks by the student requires the solution of the problem as a whole, the analysis of specific information tracking causality, i.e. transition in its research activities on a meta-subject level most satisfying the goals of formation of professional competencies.

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Моделирование процессов механических колебаний в курсе математики

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Ключевые слова и фразы: математические модели; метод Фурье; профессиональные компетенции; технологические приемы обучения.

Аннотация: В контексте компетентностного подхода предлагаются технологические приемы обучения математическому моделированию в инженерных вузах: алгоритм построения, исследования и интерпретации математических моделей и его применение в задачах о механических колебаниях.
