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NUMERICAL SOLUTION OF THE EQUILIBRIUM OF AXISYMMETRIC SOFT SHELLS*

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Abstract: We consider one axisymmetric problem of the equilibrium position of a soft rotation shell. Generalized statement of this problem is formulated in the form of variational inequality with a pseudo-monotone operator in Banach space. To solve this variational inequality, we suggest the iterative method. This method was realized numerically. The numerical experiments made for the model problems confirmed the efficiency of the iterative method.

Introduction. We consider an axisymmetric problem of the equilibrium position of a soft rotation shell. The latter is formed by two interlacing families of threads. One family has a circular direction and the other does the longitudinal one. For longitudinal threads, we assume that the dependence of the modulus of the tightening force on the degree of extension is described by a function with a power growth. For circular threads, we impose no constraints on the growth of the function which describes the dependence of the modulus of the tightening force on the degree of extension. Such problems have numerous applications [1 – 3].

Mathematically, we formulate the problem as a variational inequality with a pseudo-monotone operator over a closed convex set in a Hilbert space. Note that earlier the authors investigated the stationary problems of the soft shells theory (for infinitely long cylindrical shells and netlike ones [4], including the case when obstructions exist. For these problems formulated in the form of operator equations, variational and quasivariational inequalities, we ascertain the coercivity in the generally accepted sense [10] of operators included in equations and inequalities. This enables us to use the general results of the theory of pseudo-monotone operators in order to investigate their solvability [10].

Statement of the problem. We consider an axisymmetric equilibrium problem for a soft (i., e., immune to compressive forces) netlike rotation shell under mass and surface loads. The shell is formed by the interlacement of two families of threads with radial and longitudinal directions. We assume that the shell boundaries are fixed, the vectors of densities of the surface and mass forces lie in the radial (passing through the axis of symmetry) plane, and the shell points also move in the radial direction. We also assume that the surface load is following, i.e., it is perpendicular to the shell surface. In a strainless state, the shell surface is a cylinder with the unit radius and the length l. We take the cylindrical system of coordinates (ρ, φ, z) as the Eulerian one; in view of

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the axisymmetric property of the problem, the surface of a distorted shell is described by the coordinates in longitudinal and radial directions $z = z(s)$, $\rho = \rho(s)$, s is the Lagrange coordinate in the longitudinal direction. In a strainless state, $z_0(s) = s$, $\rho_0(s) = 1$, $0 < s < l$.

In the cylindrical system of coordinates, this problem is described by the following system of differential equations [1]:

$$\frac{d}{ds}\left(\frac{T_1(\lambda_1)}{\lambda_1}\frac{dz}{ds}\right) + q\frac{d\rho}{ds} + \tilde{f}_1 = 0, \quad \frac{d}{ds}\left(\frac{T_1(\lambda_1)}{\lambda_1}\frac{d\rho}{ds}\right) - q\frac{dz}{ds} - T_2(\lambda_2) + \tilde{f}_2 = 0, \quad 0 < s < l. \quad (1)$$

Here $\tilde{f}$, $q = \text{const}$ are the known functions which characterize the mass and surface forces, T_1, T_2 are the functions which define the dependence of the modulus of the tightening force in threads on the relative degrees of extension λ_1, λ_2 in the longitudinal and circular directions, $\lambda_1 = \left((\rho')^2 + (z')^2\right)^{1/2}$, $\lambda_2 = \rho$.

Equations (1) are supplemented with the boundary conditions

$$z(0) = 0, \quad z(l) = l, \quad \rho(0) = 1, \quad \rho(l) = 1. \quad (2)$$

In addition, the constraint $\rho(s) \ge 0$ is natural for the cylindrical system of coordinates. This inequality means to prevent self-intersection of the shell.

We formulate the variational problem which corresponds to the boundary value one (1), (2) in terms of displacements $u(s) = \left(u_1(s), u_2(s)\right)$, $u_1(s) = z(s) - s$, $u_2(s) = \rho(s) - 1$,

$$\int_0^l \frac{T_1(\lambda_1)}{\lambda_1}\left((1+u_1', u_2'), \eta'\right)ds + q\int_0^l \left[(1+u_1')\eta_2 + u_2'\eta_1'\right]ds +$$

$$+q\int_0^l \left[\frac{1}{2}u_2^2\eta_1' + (1+u_1')u_2\eta_2\right]ds + \int_0^l T_2(\lambda_2)\,\eta_2\,ds = \int_0^l \left(\tilde{f}, \eta\right)ds \quad \forall \eta \in C_0^\infty(0,l),$$

where $\lambda_1 = \left(\left((1+u_1')\right)^2 + \left(u_2'\right)^2\right)^{1/2}$, $\lambda_2 = 1 + u_2$.

Concerning the functions T_1 and T_2 we assume that

$T_1(\xi) = 0$, $T_2(\xi) = 0$, $\xi \le 1$ (the shell is immune to compressive forces), (3)

T_1, T_2 are continuous, non-decreasing, (4)

T_1 demonstrates power growth of order $p - 1 > 0$ s at infinity, i.e., there exist positive k_0, k_1 such that

$$k_0(\xi - 1)^{p-1} \le T_1 \le k_1\xi^{p-1}, \quad \xi \ge 1. \quad (5)$$

We introduce the space $V = \left[\overset{\circ}{W}{}_p^{(1)}(0,l)\right]^2$ with the norm $\|u\| = \left[\int_0^l |u'(s)|^p ds\right]^{1/p}$, and the set $K = \{u \in V : u_2(s) + 1 \ge 0 \ \forall s \in (0,l)\}$. Evidently, the set K is convex and closed. Conjugate to V is the space $V^* = \left[\overset{\circ}{W}{}^{(-1)}_{p^*}(0,l) \right]^2$, $p^* = p/(p-1)$, the duality relation between V and V^* we denote by $\langle \cdot, \cdot \rangle$.

We define the operators $A, B, D, T : V \to V^*$ and the element $f \in V^*$ using the forms

$$\langle Au, \eta \rangle = \int_0^l \frac{T_1(\lambda_1)}{\lambda_1} ((1+u_1', u_2'), \eta')\, ds;$$

$$\langle Hu, \eta \rangle = \int_0^l \left[\frac{1}{2} u_2^2\, \eta_1' + (1+u_1') u_2 \eta_2 \right] ds;$$

$$\langle B u, \eta \rangle = \int_0^l \left[(1+u_1')\eta_2 + u_2'\, \eta_1' \right] ds;$$

$$\langle Hu, \eta \rangle = \int_0^l T_2(\lambda_2)\, \eta_2\, ds;$$

$$\langle f, \eta \rangle = \int_0^l \left(\tilde{f}, \eta \right) dx.$$

The correctness of the definition of these operators follows from the continuity of T_2, the embedding of $\overset{\circ}{W}{}^{(1)}_p(0,l)$ into $C(0,l)$ and conditions (3), (5).

We understand the generalized solution of the axisymmetric problem of the equilibrium position of the soft rotation shell (the latter is fixed along the edges and experiences the mass and surface loads) as the function $u \in K$ which satisfies the variational inequality

$$\langle (A+D)u, \eta - u \rangle + q \langle (B+H)u, \eta - u \rangle \ge \langle f, \eta - u \rangle \quad \forall \eta \in K. \tag{6}$$

If conditions (3) – (5) hold then the operator A is monotone, potential, coercive and bounded, the operator B is potential, pseudo-monotone and Lipschitz-continuous with Lipschitz constant $k_2 = 2l^{2/p^*}$, the operator D is potential, compact, bounded, monotone and $\langle D\eta, \eta \rangle \ge 0$ for all $\eta \in V$, the operator H is potential, pseudo-monotone, continuous and $|\langle Hu, \eta \rangle| \le k_3 \|u\| [1 + \|u\|] \|\eta\|$ for all $u, \eta \in V$, $k_3 = 2\sqrt{2} l^{3/p^*}$ [4]. Based on these properties of the operators we prove the existence theorem for variational inequality (6).

Theorem 1. Let conditions (3) – (5) hold. Then

1) for $p > 3$ variational inequality (6) has at least one solution for any q;

2) for $p = 3$ variational inequality (6) has at least one solution for any q satisfying the inequality $|q| < k_0/k_3$;

3) for $1 < p < 3$ for any $\delta > 0$ there is $q_\delta > 0$ such that variational inequality (6) has at least one solution for any q, f satisfying the inequalities $|q| < q_\delta$, $\|f\|_{V^*} \le \delta$.

Iterative method. To solve variational inequality (6) we use the suggested and investigated in [7, 8] iterative method.

Let u^0 be an arbitrary given element. For $k=1,2,\ldots$ we find u^{k+1} as a solution of the variational inequality

$$\left\langle J(u^{k+1}-u^k), \eta-u^{k+1}\right\rangle \ge \tau\left\langle f-Pu^k, \eta-u^{k+1}\right\rangle \quad \forall\eta\in K, \tag{7}$$

where $\tau > 0$ is an iterative parameter, $P=A+D+q(B+H)$, $J: V\to V^*$ is dual operator (see [10, p. 174]) generated by some function $\Phi: [0,+\infty)\to[0,+\infty)$ such that Φ is continuous strictly monotone increasing,

$$\Phi(0)=0,\ \Phi(\xi)\to+\infty \text{ as } \xi\to+\infty. \tag{8}$$

The element u^{k+1} is uniquely determined by u^k from (7). Indeed, the dual operator is demi-continuous, strictly monotone and coercive, therefore variational inequality (7) has a unique solution.

Suppose that, in addition to (3) – (5), the functions T_1 and T_2 also satisfy the following conditions

$$\left(T_1(\xi)-T_1(\zeta)\right)/(\xi-\zeta)\le k_4(1+\xi+\zeta)^{p-2},\quad k_4>0,\quad p>1; \tag{9}$$

$$\left(T_1(\xi)-T_1(\zeta)\right)/(\xi-\zeta)\le k_5(1+\xi+\zeta)^{p-2},\quad k_5>0,\quad 2>p>1 \tag{10}$$

and

$$\left(T_2(\xi)-T_2(\zeta)\right)/(\xi-\zeta)\le k_6(1+\xi+\zeta)^{\sigma-1},\quad k_6>0,\quad \sigma>1. \tag{11}$$

We say that the operator A satisfies the Lipschitz-type bounded continuity condition if

$$\| Au-A\eta\|_{V^*}\le \mu_A(R)\,\Phi_A(\|u-\eta\|)\quad \forall\, u,\ \eta\in V,\quad R=\max\{\|u\|, \|\eta\|\}, \tag{12}$$

where $\mu_A: [0,+\infty)\to[0,+\infty)$ is non decreasing function, Φ_A satisfies the condition (8).

Recall that the operator A is called inversely strongly monotone if

$$\| Au-A\eta\|_{V^*}^2\le d\left\langle Au-A\eta, u-\eta\right\rangle\quad \forall\, u,\ \eta\in V,\quad d>0. \tag{13}$$

The following results are valid.

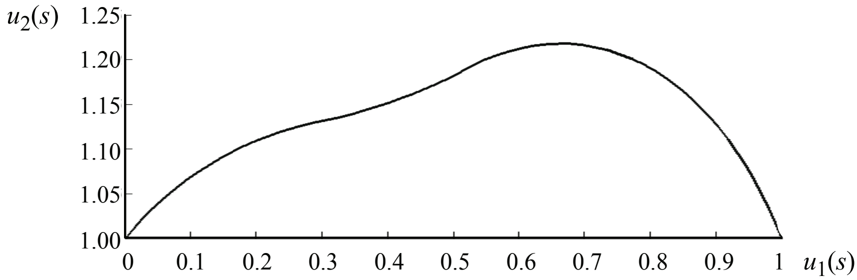
Theorem 2. Suppose that K is a nonempty closed convex subset of a reflexive Banach space V with a strictly convex conjugate V^*, the operator P is pseudomonotone, coercive, potential with the potential F and bounded Lipschitz-continuous with functions Φ and μ. Suppose further that the condition

$$0<\tau<\min\left\{1, \frac{1}{\mu_0}\right\},\ \mu_0=\mu\left(R_0+\Phi^{-1}(R_1)\right),\ R_0=\sup_{u\in S_0}\|u\|,\ R_1=\sup_{u\in S_0}\|Pu-f\|_{V^*}$$

holds, where $S_0=\left\{u\in K: F(u)\le F(u^0)\right\}$. Then the constructed by (7) iterative sequence $\{u^k\}_k$ is bounded and all its weak limit points are solutions of the variational inequality

$$\left\langle Pu, \eta-u\right\rangle\ge\left\langle f, \eta-u\right\rangle\quad \forall\eta\in K. \tag{14}$$

Theorem 3. Suppose that K is a nonempty closed convex subset of a Hilbert space V, the operator P is inversely strongly monotone, coercive, potential with the potential F. Suppose further that the condition $0<\tau<2/d$ holds. Then the constructed by (7)



Generatrix of the deformed shell

iterative sequence $\{u^k\}_k$ converges weakly to some solution of variational inequality (14).

If the functions T_1 and T_2 satisfy the conditions $(3)-(5)$, $(9)-(11)$ then the operator A is bounded Lipschitz-continuous with the functions $\Phi_A(\xi)=\xi$ and $\mu_A(\xi)=k_7\left(3l^{1/p}+2\xi\right)^{p-2}$, $k_7=\max\{k_1,k_4\}$ for $p\geq 2$ and with the functions $\Phi_A(\xi)=\xi^{p-1}$ and $\mu_A(\xi)=k_8=\max\{2k_1,k_5\}$ for $2>p>1$; the operator D is bounded Lipschitz-continuous with the functions $\Phi_D(\xi)=\xi$ and $\mu_D(\xi)=k_9\left(3+2l^{1/p^*}\xi\right)^{\sigma-1}$, $k_9=k_6\,l^{2/p^*+1}$ for $p>2$ and with the functions $\Phi_D(\xi)=\xi^{p-1}$ and $\mu_D(\xi)=k_9 2^{2-p}$ for $2>p>1$. If $p=2$ and the condition (11) with $\sigma=1$ holds then the operator D satisfies the inequality (13) with $d=k_6/2$.

Thus, for the considered problem the theorems 2, 3 can be applied. It was developed the software, using MATLAB environment. Numerical experiments for model problems are performed. The numerical results are presented in Figure, which shows the generatrix of the deformed shell. Calculations were performed with the following inputs. Functions $T_1(\xi)=T_2(\xi)=\xi$ for $\xi\geq 1$, $f_1(s)\equiv 0$, $f_2(s)=0.005$, $0<s<0.3$, $f_2(s)=0$, $0.3\leq s\leq 0.5$, $f_2(s)=0.01$, $0.5<s<l=1$, $q=0.001$.

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Численное решение задач о равновесии осесимметричных мягких оболочек

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Ключевые слова: итерационный метод; математическое моделирование; мягкая оболочка; положение равновесия; псевдомонотонный оператор; численный эксперимент.

Аннотация: Рассмотрена осесимметричная задача об определении положения равновесия мягкой оболочки вращения. Обобщенная постановка задачи сформулирована в виде вариационного неравенства с псевдомонотонным оператором в банаховом пространстве. Для решения вариационного неравенства предложен итерационный метод, который реализован численно. Результаты численных экспериментов подтвердили эффективность предложенного итерационного метода.

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Numerische Lösung der Aufgaben über das Gleichgewicht der achsensymmetrischen weichen Hüllen

Zusammenfassung: Es ist die achsensymmetrische Aufgabe über die Bestimmung der Lage des Gleichgewichtes der weichen Hülle des Drehens betrachtet. Die verallgemeinerte Aufgabenstellung ist in Form von der Variationsungleichheit mit dem pseudomonotonen Operator in dem Banachraum abgefasst. Für die Lösung der Variationsungleichheit ist die iterative Methode angeboten. Diese Methode ist numerisch realisiert. Die Ergebnisse der numerischen Experimente haben die Effektivität der angebotenen iterativen Methode bestätigt.

Solition numérique des problèmes sur l'équilibre des envelopes axisymétriques souples

Résumé: Est examiné le problème axisymétrique sur définition de la disposition de l'équilibre de l'envelope souple de rotation. Le problème général est formulé en vue de l'inégalité variotionnelle avec un opérateur pseudomonotone dans l'espace de Banach. Pour la solution de l'inégalité variotionnelle est proposée la méthode itérative. Cette méthode est réalisée de la manière numérique. Les résultats des expériments ont confirmé l'efficacité de la méthode itérative proposée.

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